1. (Stewart 16.5/5) Find the curl and the divergence of the following vector field.

$$\underline{F}(x,y,z) := \frac{\sqrt{x}}{1+z}\underline{i} + \frac{\sqrt{y}}{1+x}\underline{j} + \frac{\sqrt{z}}{1+y}\underline{k}$$

$$\underline{F}(x,y,z) = \left(\frac{\sqrt{x}}{1+z}, \frac{\sqrt{y}}{1+x}, \frac{\sqrt{z}}{1+y}\right)$$

$$\operatorname{curl} \underline{F}(x,y,z) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial_1 & \partial_2 & \partial_3 \\ F_1 & F_2 & F_3 \end{vmatrix} \underset{\substack{\uparrow \\ \text{formal} \\ \text{computation}}}{=} \left(-\frac{\sqrt{z}}{(1+y)^2} - 0, -\frac{\sqrt{x}}{(1+z)^2} - 0, -\frac{\sqrt{y}}{(1+x)^2} - 0\right) = -\left(\frac{\sqrt{z}}{(1+y)^2}, \frac{\sqrt{x}}{(1+z)^2}, \frac{\sqrt{y}}{(1+x)^2}\right)$$

$$\operatorname{div} \underline{F}(x,y,z) = \partial_1 F_1(x,y,z) + \partial_2 F_2(x,y,z) + \partial_3 F_3(x,y,z) = -\frac{1}{2}\left(\frac{1}{\sqrt{x}(1+z)} + \frac{1}{\sqrt{y}(1+x)} + \frac{1}{\sqrt{z}(1+y)}\right)$$

2. (Stewart 16.5/19) Determine whether or not the vector field is conservative.

$$\underline{F}(x,y,z) := yz^2e^{xz}\underline{i} + ze^{xz}\underline{j} + xyze^{xz}\underline{k}$$

$$\operatorname{curl} \underline{F}(x,y,z) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial_1 & \partial_2 & \partial_3 \\ F_1 & F_2 & F_3 \end{vmatrix} = \Big(\underbrace{xz\,e^{xz} - e^{xz}}_{\neq 0}, \ldots, \ldots\Big) \Rightarrow \underline{F} \text{ is not conservative}$$

3. (Stewart 16.5/23) Show that any vector field of the form

$$\underline{F}(x,y,z) := f(x)\underline{i} + g(y)\underline{j} + h(z)\underline{k}$$

is irrotational.

$$\operatorname{curl} \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial_1 & \partial_2 & \partial_3 \\ F_1 & F_2 & F_3 \end{vmatrix} = \Big(\overbrace{\tfrac{\partial}{\partial y} h(z)}^{0} - \overbrace{\tfrac{\partial}{\partial z} g(y)}^{0}, \overbrace{\tfrac{\partial}{\partial z} f(x)}^{0} - \overbrace{\tfrac{\partial}{\partial x} h(z)}^{0}, \overbrace{\tfrac{\partial}{\partial x} g(y)}^{0} - \overbrace{\tfrac{\partial}{\partial y} f(x)}^{0}\Big) = (0,0,0)$$

$$\Rightarrow \underline{F} \text{ is irrotational}$$

4. (Stewart 16.5/29) Prove the identity, assuming that the appropriate partial derivatives exist and are continuous. ($\underline{F}, \underline{G} : \mathbb{R}^3 \rightarrowtail \mathbb{R}^3$)

$$\operatorname{div}(\underline{F} \times \underline{G}) = \underline{G} \cdot \operatorname{curl} \underline{F} - \underline{F} \cdot \operatorname{curl} \underline{G}$$

$$\operatorname{div} \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ F_1 & F_2 & F_3 \\ G_1 & G_2 & G_3 \end{vmatrix} = \operatorname{div}\left(F_2G_3 - F_3G_2, F_3G_1 - F_1G_3, F_1G_2 - F_2G_1\right) =$$

$$= (\partial_1 F_2)G_3 + F_2(\partial_1 G_3) - (\partial_1 F_3)G_2 - F_3(\partial_1 G_2)$$

$$+ (\partial_2 F_3)G_1 + F_3(\partial_2 G_1) - (\partial_2 F_1)G_3 - F_1(\partial_2 G_3)$$

$$+ (\partial_3 F_1)G_2 + F_1(\partial_3 G_2) - (\partial_3 F_2)G_1 - F_2(\partial_3 G_1) = \boxed{\underline{G} \cdot \operatorname{curl}(\underline{F})}\ \boxed{- \underline{F} \cdot \operatorname{curl}(\underline{G})}$$

5.

35. Use Green's Theorem in the form of Equation 13 to prove **Green's first identity**:

$$\iint_D f\nabla^2 g \, dA = \oint_C f(\nabla g) \cdot \mathbf{n} \, ds - \iint_D \nabla f \cdot \nabla g \, dA$$

where D and C satisfy the hypotheses of Green's Theorem and the appropriate partial derivatives of f and g exist and are continuous. (The quantity $\nabla g \cdot \mathbf{n} = D_{\mathbf{n}}\, g$ occurs in the line integral; it is the directional derivative in the direction of the normal vector $\mathbf{n}$ and is called the **normal derivative** of g.)

If $f : \mathbb{R}^3 \to \mathbb{R}$, $\underline{F} : \mathbb{R}^3 \to \mathbb{R}^3$ are differentiable:

$$\operatorname{div}(f \cdot \underline{F}) = (\partial_1 f)F_1 + f(\partial_1 F_1) + (\partial_2 f)F_2 + f(\partial_2 F_2) + (\partial_3 f)F_3 + f(\partial_3 F_3) = \nabla f \cdot \underline{F} + f \cdot \operatorname{div}(\underline{F})$$

If $f: \mathbb{R}^3 \to \mathbb{R}$, $g: \mathbb{R}^3 \to \mathbb{R}$ are twice differentiable

$$\operatorname{div}(f \cdot \nabla g) = \nabla f \cdot \nabla g + f \underbrace{\operatorname{div}(\nabla g)}_{\nabla^2 g}$$

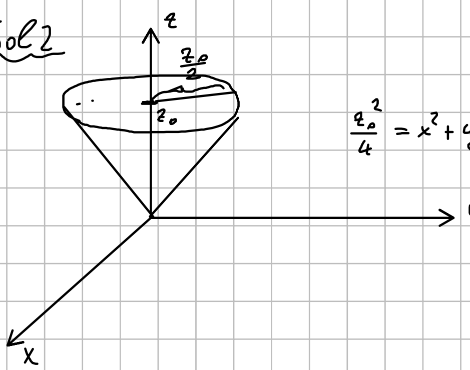
$$\iint_D f \nabla^2 g \, dA = \iint_D \operatorname{div}(f \nabla g)\, dA - \iint_D \nabla f \cdot \nabla g \, dA = \int_C f(\nabla g) \cdot \underline{n} \, d\underline{r} - \iint_D \nabla f \cdot \nabla g \, dA$$

by Green's first identity

6. (Stewart 16.6/ Example 7) Find a parametric representation for the surface $z = 2\sqrt{x^2+y^2}$, that is, the top half of the cone $z^2 = 4x^2 + 4y^2$.

Sol1 $\quad (x, y, 2\sqrt{x^2+y^2}) \quad (x \in \mathbb{R}, y \in \mathbb{R})$

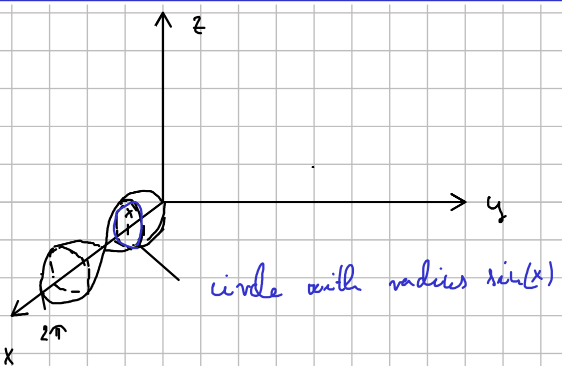
Sol2



$$\frac{z_0^2}{4} = x^2 + y^2 \implies \text{radius} = \frac{z_0}{2}$$

$\implies$ Parametric representation: $\{(r\cos(\vartheta), r\sin(\vartheta), 2r) \mid r \geq 0,\ 0 \leq \vartheta \leq 2\pi\}$

7. (Stewart 16.6/ Example 8) Find parametric equations for the surface generated by rotating the curve $y = \sin(x)$, $0 \leq x \leq 2\pi$, about the x-axis. (Use these equations to graph the surface of revolution.)



Parametrization: $(x, \sin(x)\cos(\vartheta), \sin(x)\cdot\sin(\vartheta))$ $\quad x \in [0, 2\pi]$, $\vartheta \in [0, 2\pi]$

8. (Stewart 16.6/ 33) Find an equation of the tangent plane to the parametric surface

$$x = u + v, \quad y = 3u^2, \quad z = u - v$$

at the point $(2, 3, 0)$.

$$\underline{r}(u,v) = (u+v, 3u^2, u-v)$$

$\underline{r}(u,v) = \underline{(2,3,0)} \implies u = v = 1$ $\quad$ normal vector of the tangent plane: $\partial_1 \underline{r}(1,1) \times \partial_2 \underline{r}(1,1)$

$\partial_1 \underline{r}(u,v) = (1, 6u, 1)$ $\quad \partial_2 \underline{r}(u,v) = (1, 0, -1)$

$$\partial_1 \underline{r}(1,1) \times \partial_2 \underline{r}(1,1) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 6 & 1 \\ 1 & 0 & -1 \end{vmatrix} = \underline{(-6, 2, -6)}$$

expand along the first row

eq. of the tangent plane:

$$-6(x-2) + 2(y-3) - 6z = 0$$

$$-6x + 2y - 6z = -6$$

$$\iff \boxed{3x - y + 3z = 3}$$

9. (Stewart 16.6/ 41,45) Find the area of the surface.

a) The part of the plane $x + 2y + 3z = 1$ that lies inside the cylinder $x^2 + y^2 = 3$

b) The part of the surface $z = xy$ that lies within the cylinder $x^2 + y^2 = 1$.

a) Sol 1 $\quad z = \frac{1}{3} - \frac{x}{3} - \frac{2}{3}y$, $\quad x^2+y^2 = 3$

parametrization: $\left(r\cos(\vartheta), r\sin(\vartheta), \frac{1}{3}\left(1 - r\cos(\vartheta) - 2r\sin(\vartheta)\right)\right) = \mathbb{p}(r,\vartheta)$

$(r \in [0,\sqrt{3}], \ \vartheta \in [0,2\pi])$

$$\partial_1 \mathbb{p}(r,\vartheta) = \left(\cos(\vartheta), \sin(\vartheta), \frac{1}{3}\left(-\cos(\vartheta) - 2\sin(\vartheta)\right)\right)$$

$$\partial_2 \mathbb{p}(r,\vartheta) = \left(-r\sin(\vartheta), r\cos(\vartheta), \frac{1}{3}\left(r\sin(\vartheta) - 2r\cos(\vartheta)\right)\right)$$

$$\partial_1 \mathbb{p}(r,\vartheta) \times \partial_2 \mathbb{p}(r,\vartheta) = \begin{pmatrix} \frac{1}{3}\left(r\sin^2(\vartheta) - 2r\sin(\vartheta)\cos(\vartheta) + r\cos^2(\vartheta) + 2r\sin(\vartheta)\cos(\vartheta)\right) \\ \frac{1}{3}\left(r\sin(\vartheta)\cos(\vartheta) + 2r\sin^2(\vartheta) - r\sin(\vartheta)\cos(\vartheta) + 2r\cos^2(\vartheta)\right) \\ r\cos^2(\vartheta) + r\sin^2(\vartheta) \end{pmatrix} = \begin{pmatrix} \frac{r}{3} \\ \frac{2r}{3} \\ r \end{pmatrix} = r\begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ 1 \end{pmatrix}$$

$$A(S) = \int_S 1 = \int_0^{\sqrt{3}}\int_0^{2\pi} \|\partial_1\mathbb{p}(r,\vartheta)\times\partial_2\mathbb{p}(r,\vartheta)\| \, d\vartheta \, dr = \int_0^{\sqrt{3}}\int_0^{2\pi} \frac{r}{3}\sqrt{1+4+9} \, d\vartheta \, dr = \frac{2\pi}{3}\cdot\sqrt{14}\left[\frac{r^2}{2}\right]_{r=0}^{\sqrt{3}} = \sqrt{14}\cdot\pi$$

Sol 2: $\quad z = \frac{1-x-2y}{3} = g(x,y) \qquad \partial_1 g(x,y) = -\frac{1}{3} \qquad \partial_2 g(x,y) = -\frac{2}{3}$

$D := \{(x,y)\in\mathbb{R}^2 \mid x^2+y^2 \leq 3\}$

$S = \{(x,y,g(x,y)) \mid (x,y)\in D\} \implies A(S) = \int_D \sqrt{1+(\partial_1 g(x,y))^2 + (\partial_2 g(x,y))^2} \, d(x,y) = \int_D \frac{1}{3}\sqrt{9+1+4} \, d(x,y) =$

$= \frac{\sqrt{14}}{3} \underbrace{A(D)}_{3\pi} = \sqrt{14}\,\pi$

b) $\quad z = xy = g(x,y) \qquad \partial_1 g(x,y) = y \qquad \partial_2 g(x,y) = x$

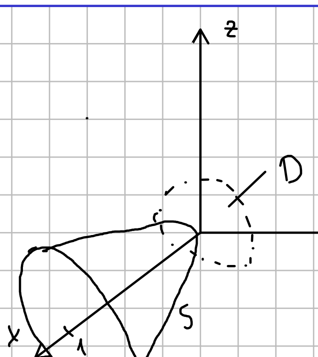
$x^2+y^2 = 1$

$$A(S) = \int_D \sqrt{1+(\partial_1 g(x,y))^2 + (\partial_2 g(x,y))^2} \, d(x,y) = \int_D \sqrt{1+y^2+x^2} \, dx \underset{\text{polar}}{=} \int_0^1\int_0^{2\pi} \sqrt{1+r^2}\cdot r \, d\vartheta \, dr =$$

$D = \{(x,y)\in\mathbb{R} \mid x^2+y^2 \leq 1\}$

$$= \frac{2\pi}{2}\cdot\int_0^1 \sqrt{1+r^2}\cdot 2r \, dr = \pi\left[\frac{2}{3}(1+r^2)^{\frac{3}{2}}\right]_{r=0}^{1} = \frac{2\pi}{3}\left(2\cdot\sqrt{2} - 1\right)$$

10. (Stewart 16.7/ 13) Evaluate the surface integral $\iint_S z^2 \, dS$, where S is the part of the paraboloid $x = y^2 + z^2$ given by $0 \leq x \leq 1$.



$g(y,z) = y^2+z^2 \qquad \partial_1 g(y,z) = 2y \qquad \partial_2 g(y,z) = 2z$

$D := \{(y,z)\in\mathbb{R}^2 \mid y^2+z^2 \leq 1\}$

$$\iint_S z^2 = \int_D z^2 \sqrt{1+(\partial_1 g(y,z))^2 + (\partial_2 g(y,z))^2} \, d(x,y) = \int_D z^2\sqrt{1+4y^2+4z^2} \, d(x,y) =$$

$$\underset{\text{polar}}{=} \int_0^1\int_0^{2\pi} r^3\cdot\sin^2(\vartheta)\sqrt{1+4r^2} \, d\vartheta \, dr = \int_0^1 r^3\sqrt{1+4r^2} \, dr \cdot \int_0^{2\pi} \sin^2(\vartheta) \, d\vartheta = \circledast$$

$$\int_0^{2\pi} \underbrace{\sin^2(\theta)}_{\frac{1-\cos(2\theta)}{2}} d\theta = \left[\frac{\theta}{2} - \frac{\sin(2\theta)}{2}\right]_{\theta=0}^{2\pi} = \pi$$

$$\int_0^1 r^3 \sqrt{1+4r^2}\, dr = \int_1^5 \frac{t-1}{4} \sqrt{t} \cdot \frac{1}{8} dt = \frac{1}{32} \int_1^5 t^{\frac{3}{2}} - t^{\frac{1}{2}} dt = \frac{1}{32}\left[\frac{2}{5} t^{\frac{5}{2}} - \frac{2}{3} t^{\frac{3}{2}}\right]_{t=1}^{5} =$$

$$t = 1 + 4r^2 \Rightarrow r^2 = \frac{t-1}{4}$$

$$dt = 8r\,dr$$

$$= \frac{1}{32}\left(2 \cdot 5\sqrt{5} - \frac{2}{3} 5\sqrt{5} - \frac{2}{5} + \frac{2}{3}\right) =$$

$$= \frac{1}{16}\left(\frac{10}{3}\sqrt{5} + \frac{2}{15}\right) = \frac{1}{120}\left(1 + 25\sqrt{5}\right)$$

$\Rightarrow$ Answer: $(*) = \frac{\pi}{120}(1 + 25\sqrt{5})$